Operator of fractional derivative in the complex plane

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July 15, 1997

(To be published in Communications in Mathematical Physics)

Abstract

The paper deals with a fractional derivative introduced by means of the Fourier transform. The explicit form of the kernel of general derivative operator acting on the functions analytic on a curve in complex plane is deduced and the correspondence with some well known approaches is shown. In particular, it is shown how the uniqueness of the operation depends on the derivative order type (integer, rational, irrational, complex) and the number of poles of considered function in the complex plane.

1 Introduction

The fractional differentiation and integration (also called fractional calculus) is a notion almost equally old as the ordinary differential and integral calculus. Naturally, each a bit more gifted student who has just understood what is the first and second derivative can ask the question: *well, but what is for example 1.5 - fold derivative?* There are several ways how to answer such a question. An excellent review on the theory of arbitrary order differentiation (generally of complex order) including also interesting historical notes and comprehensive list of references to original papers (more than thousand items) is given in the recently published monograph [8]. Some new results were presented also in the recent conference [5] dedicated to this topic. The fractional calculus has also plenty of applications, see also e.g. [6],[4],[9] and citations therein. Possible use in quantum mechanics and the field theory is discussed in [10]. Recently, fractional derivative was mentioned also in [2] as a particular case of pseudo - differential operators applied in non - local field theory.

Apparently, the general prescription for a definition of fractional derivative is using of some representation of ordinary *n-fold* derivative (primitive function) which can be in some natural way interpolated to *n-non integer.* Actually, following the mentioned monograph, all the known approaches are always somehow connected with some of the following relations.

1) The well known formula for *n-fold* integral

$$\int_a^{x_1} dx_2 \int_a^{x_2} dx_3 ... \int_a^{x_n} f(t)dt = \frac{1}{\Gamma(n)} \int_a^{x_1} (x_1 - t)^{n-1} f(t)dt \tag{1.1}$$

allows the substitution of n by some real $\alpha > 0$. In this way fractional integration is introduced. Then fractional derivatives can be obtained by ordinary differentiation of fractional integrals. This is the basis of the construction known as Riemann - Liouville fractional calculus. Let us note, that in this approach resulting function even in the case of fractional derivatives in general depends on the fixing of the integration limit a on the right hand side of (1.1).

2) The Cauchy formula for analytic functions in some region of the complex plane

$$f^{(n)}(z_0) = \frac{\Gamma(n+1)}{2i\pi} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \tag{1.2}$$

in principle enables generalization to fractional derivatives, nevertheless the direct extension to *non-integer* values of n leads to difficulties arising from multivaluedness of the term $(z - z_0)^{\alpha+1}$ and the result also depends on the choice of the cut and integration curve .

3) Analytic continuation of derivative (integral) of exponential and power function

$$\frac{d^\alpha}{dz^\alpha} \exp(cz) = c^\alpha \exp(cz), \qquad \frac{d^\alpha}{dz^\alpha}(z-c)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(z-c)^{\beta-\alpha}. \tag{1.3}$$

Obviously these relations allow to define fractional derivatives of the functions which can be expressed as linear combinations of power and exponential functions. Also this approach is not completely consistent, as can be illustrated by fractional derivative of exponential function expanded to the power series

$$\exp(cz) = \sum_{k=0}^{\infty} \frac{(cz)^k}{\Gamma(k+1)}, \tag{1.4}$$

but for $\alpha -$ *non integer*

$$\frac{d^\alpha}{dz^\alpha} \exp(cz) = c^\alpha \exp(cz) \neq c^\alpha \sum_{k=0}^{\infty} \frac{(cz)^{k-\alpha}}{\Gamma(k-\alpha+1)} = \frac{d^\alpha}{dz^\alpha} \left(\sum_{k=0}^{\infty} \frac{(cz)^k}{\Gamma(k+1)} \right). \tag{1.5}$$

So the task of extrapolation of integer derivative order to arbitrary one has not the unique solution.

The approach proposed in this paper is based on the fractional derivative of the exponential function (1.3) entering the Fourier transform of a given function. On this basis in Sec.2 explicit form of the kernel of fractional derivative operator is deduced. In Sec.3 the composition relation for the derivative operator is proved. The generalization of the case of the functions on the real axis to the case of function on the complex plane is done in Sec.4, which is concluded by theorem summarizing the results. The last section is devoted to the discussion of some consequences following from the theorem and to a comparison with the known approaches as well.

2 Definition of fractional derivative by means of Fourier transform

Let $f(x)$ be a function having Fourier picture $\widetilde{f}(k)$:

$$\widetilde{f}(k) = \int_{-\infty}^{+\infty} f(x)\exp(ikx)dx, \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widetilde{f}(k)\exp(-ikx)dk. \tag{2.1}$$

Then let us create the function

$$f^{\alpha}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (-ik)^{\alpha}\widetilde{f}(k)\exp(-ikx)dk, \qquad \alpha > -1 \tag{2.2}$$

and define

$$D^{\alpha}(w) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (-ik)^{\alpha}\exp(-ikw)dk. \tag{2.3}$$

The function $f^{\alpha}(x)$ can be formally expressed

$$f^{\alpha}(x) = \mathbf{D}^{\alpha}f = \int_{-\infty}^{+\infty} D^{\alpha}(x-y)f(y)dy. \tag{2.4}$$

Now let us calculate the integral (2.3), which depends on the way of passing about the singularity $k=0$ and choice of the branch and cut orientation of the function k^{α}. For the beginning let us assume the cut is given by the half line either $(0,-\infty)$ or $(0,+\infty)$. For complex functions $\xi^{\alpha} = (\xi_1 + i\xi_2)^{\alpha}$ we shall accept the phase convention

$$\lim_{\xi_2 \to 0+} (\xi_1 + i\xi_2)^{\alpha} = \mid \xi_1^{\alpha} \mid \qquad \xi_1, \xi_2 \geq 0, \tag{2.5}$$

i.e. it holds

$$\begin{array}{llccc}
 & & \xi < 0 & \xi \geq 0 & \text{cut orientation} \\
\lim\limits_{\epsilon \to 0+} (\xi + i\epsilon)^{\alpha}/ \mid \xi^{\alpha} \mid = & \left\{ \begin{array}{l} \\ \\ \end{array} \right. & \begin{array}{c} \exp(+i\pi\alpha) \\ \exp(+i\pi\alpha) \end{array} & \begin{array}{c} 1 \\ 1 \end{array} & \begin{array}{c} (0,+\infty) \\ (0,-\infty) \end{array} \\
\lim\limits_{\epsilon \to 0+} (\xi - i\epsilon)^{\alpha}/ \mid \xi^{\alpha} \mid = & \left\{ \begin{array}{l} \\ \\ \end{array} \right. & \begin{array}{c} \exp(+i\pi\alpha) \\ \exp(-i\pi\alpha) \end{array} & \begin{array}{c} \exp(2i\pi\alpha) \\ 1 \end{array} & \begin{array}{c} (0,+\infty) \\ (0,-\infty) \end{array}
\end{array} \tag{2.6}$$

Let us define

$$D^{\alpha}_{\pm}(w) = \frac{(-1)^{\alpha}}{2\pi} \int_{-\infty \pm i0}^{+\infty \pm i0} (ik)^{\alpha} \exp(-ikw) dk. \tag{2.7}$$

If we accept phase convention (2.6) for k^{α} in the integral (2.7), then the uncertainty of phase of the expression (2.7) is involved only in the factor

$$(-1)^{\alpha} = \exp(i\alpha \left[2n+1\right] \pi), \tag{2.8}$$

where n is any integer number. Within arbitrariness given by (2.8) the functions D^{α}_{+}, D^{α}_{-} do not depend on the cut orientation and it holds

$$\begin{aligned} D^{\alpha}_{+}(w) = 0 \qquad & \text{for } w < 0 \\ D^{\alpha}_{-}(w) = 0 \qquad & \text{for } w > 0, \end{aligned} \tag{2.9}$$

which is evident from the corresponding integrals having the paths closed in infinity - for $D^{\alpha}_{+}(D^{\alpha}_{-})$ in upper (lower) half-plane. Now we shall calculate the integrals (2.7) in remaining regions of w. Let us split them into two parts

$$D^{\alpha}_{\pm}(w) = \frac{(-1)^{\alpha}}{2\pi} \left[\int_{-\infty \pm i0}^{0 \pm i0} (ik)^{\alpha} \exp(-ikw) dk + \int_{0 \pm i0}^{+\infty \pm i0} (ik)^{\alpha} \exp(-ikw) dk \right] \tag{2.10}$$

and substitute real parameter w by complex one

$$\begin{aligned} z_1 = w + i\epsilon \qquad & \text{for } k < 0 \\ z_2 = w - i\epsilon \qquad & \text{for } k > 0, \end{aligned} \tag{2.11}$$

where $\epsilon > 0$. This substitution ensures absolute convergence of both integrals in (2.10). Next let us make in (2.10) further substitution

$$\begin{aligned} \xi = ikz_1 \qquad & \text{for } k < 0 \\ \xi = ikz_2 \qquad & \text{for } k > 0. \end{aligned} \tag{2.12}$$

In this way instead of $D^{\alpha}_{\pm}(w)$ we get functions depending also on ϵ :

$$E^{\alpha}_{\pm}(w, \epsilon) = \frac{(-1)^{\alpha}}{2\pi i} \left[\frac{1}{z_1^{\alpha+1}} \int_{K_1} \xi^{\alpha} \exp(-\xi) d\xi + \frac{1}{z_2^{\alpha+1}} \int_{K_2} \xi^{\alpha} \exp(-\xi) d\xi \right]. \tag{2.13}$$

We assume

a) Cut of complex function k^{α} is given by half-line (0,$+\infty$).

b) Function values z_1^{α}, z_2^{α} are considered values of one complex function z^{α} in the two different points. According to the cut orientation of z^{α} one can get either $z_2^{\alpha} = (\ z_1^{\alpha})^{*}$ or $z_2^{\alpha} = (\ z_1^{\alpha})^{*} \exp(2i\pi\alpha)$. Cut of the function z^{α} is assumed

$$\begin{aligned} (0, +\infty) \qquad & \text{for } E^{\alpha}_{+} \\ (0, -\infty) \qquad & \text{for } E^{\alpha}_{-}. \end{aligned} \tag{2.14}$$

integral	**variable**	$k<0$	$k>0$
	k	π	0
E_{+}^{α}	z	$(0,\pi)$	$(\pi,2\pi)$
	ξ	$(3\pi/2,5\pi/2)$	$(3\pi/2,5\pi/2)$
	k	π	2π
E_{-}^{α}	z	$(0,\pi)$	$(-\pi,0)$
	ξ	$(3\pi/2,5\pi/2)$	$(3\pi/2,5\pi/2)$

Table 1: The phase correspondence of variables ξ, z, k depending on the cut orientation

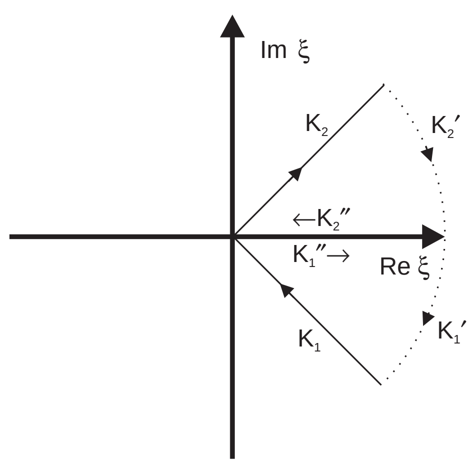

Figure 1: Integration paths in Eqs. (2.13),(2.15).

Latter on we shall come back to these assumptions and judge how they affected our result.

Assumptions concerning the cuts of k^α, z^α correspond to phases or to the intervals of phases of variables k, z and correspondingly to phases of the variable ξ. All considered possibilities are summarized in Tab.1 . The corresponding integration paths are shown in Fig.1. Instead of the interval $(3\pi/2, 5\pi/2)$ for $\arg\xi$ we took the interval $(-\pi/2, \pi/2)$ since the corresponding phase shift is the same for both integrals in (2.13) and can be included in factor the $(-1)^\alpha$ ahead of the integrals. Obviously, it holds

$$\int_{K_2} \xi^\alpha \exp(-\xi) d\xi = -\int_{K_1} \xi^\alpha \exp(-\xi) d\xi = \int_0^\infty \xi^\alpha \exp(-\xi) d\xi = \Gamma(\alpha+1). \tag{2.15}$$

After inserting into (2.13) we get

$$E_\pm^\alpha(w,\epsilon) = \frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi}\left(\frac{1}{(w+i\epsilon)^{\alpha+1}} - \frac{1}{(w-i\epsilon)^{\alpha+1}}\right), \qquad \epsilon > 0. \tag{2.16}$$

We shall regard the integrals (2.7) as the limits

$$D_\pm^\alpha(w) = \lim_{\epsilon\to 0+} E_\pm^\alpha(w,\epsilon), \tag{2.17}$$

hence the kernel of the operator (2.3) which can be also considered the generalized function is symbolically written

$$D_\pm^\alpha(w) = \frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi}\left(\frac{1}{(w+i0)^{\alpha+1}} - \frac{1}{(w-i0)^{\alpha+1}}\right), \tag{2.18}$$

where the two modes correspond to different cuts of $(w+i\epsilon)^{\alpha+1}$

$$\begin{array}{ll} D_+^\alpha(w) & \text{for cut } (0,+\infty) \\ D_-^\alpha(w) & \text{for cut } (0,-\infty). \end{array} \tag{2.19}$$

Let us note the function (2.18) is well defined for any complex $\alpha = \alpha_1 + i\alpha_2 \neq -1, -2, ..$, but for the beginning we assume $\alpha_2 = 0$. Now let us go back to the assumptions *a*), *b*) which imply result (2.16).

a)Assuming the opposite cut orientation $(0, -\infty)$ for k^α in Eq. (2.10) and repeating the corresponding sequence of steps does not change anything for $E_+^\alpha(w,\epsilon)$ whereas in the case of $E_-^\alpha(w,\epsilon)$ the phase $\arg\xi$ will shift by -2π in both integrals in (2.13), but this change can be included in factor $(-1)^\alpha$ ahead of the integral.

b) Let us assume the cuts in z^α having the opposite orientation than in (2.14). Let us take e.g. function $E_+^\alpha(w,\epsilon)$, then the phase of ξ complies with

$$\begin{array}{rclll} 3\pi/2 & < \arg\xi < & 5\pi/2 & & \text{for } k > 0 \\ -\pi/2 & < \arg\xi < & \pi/2 & & \text{for } k < 0, \end{array} \tag{2.20}$$

i.e. the phase of second integral in (2.13) is now shifted by $-2\pi\alpha$ and instead of (2.16) we get

$$E_+^\alpha(w,\epsilon) = \frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi}\left(\frac{1}{(w+i\epsilon)^{\alpha+1}} - \frac{\exp(-2i\pi\alpha)}{(w-i\epsilon)^{\alpha+1}}\right), \qquad \epsilon > 0. \tag{2.21}$$

Obviously this function for $w \in (-\infty, +\infty)$ with assumed cut of z^α on $(0, -\infty)$ is identically equal to the function (2.16) with the cut on $(0, +\infty)$. The analogous result can be obtained also for E_-^α.

So we have shown, that result (2.16) of the integration (2.7) does not depend on the choice of cut orientation of the functions k^α and w^α. The cut orientation of w^α in (2.16) and (2.18) is dictated only by the way of passing around the singularity $k = 0$ in initial integrals (2.7) and the correspondence (2.19) always holds. Next let us notice the important property of the functions $D_\pm^\alpha$, $E_\pm^\alpha$. Obviously it holds

$$\begin{aligned} \frac{d}{dw} E_\pm^\alpha(w, \epsilon) &= E_\pm^{\alpha+1}(w, \epsilon), \\ \frac{d}{dw} D_\pm^\alpha(w) &= D_\pm^{\alpha+1}(w). \end{aligned} \tag{2.22}$$

Now let us go back to the Eq. (2.4) and consider how to calculate this integral. It is possible either

a) First to make the difference of both terms in (2.16), then integration and limit for $\epsilon \to 0$,

or

b) First to calculate both integrals independently, then make the limit of their difference.

Let us discuss the both ways and compare the results.

a) *The action of operator* $\mathbf{D}_\pm^\alpha$ *as the integral of difference*

We make calculation separately for the two cases:

*a*1) $\alpha = n \geq 0$ *is an integer number*

In that case the complex function w^α has no cuts (we can omit the subscript $\pm$) and for $n = 0$ we can write

$$E^0(w, \epsilon) = \frac{1}{\pi} \cdot \frac{\epsilon}{w^2 + \epsilon^2}, \tag{2.23}$$

i.e. for $\epsilon \to 0$ we get the known representation of δ-function (see e.g. [3],p.35)

$$\delta(w) = \lim_{\epsilon \to 0+} \frac{1}{\pi} \cdot \frac{\epsilon}{w^2 + \epsilon^2} \tag{2.24}$$

which acts on the function f,

$$\int_{-\infty}^{+\infty} \delta(x - y) f(y) dy = \lim_{\epsilon \to 0+} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\epsilon}{(x - y)^2 + \epsilon^2} f(y) dy = f(x). \tag{2.25}$$

Using equations (2.22)$-$(2.25) one can easily show

$$\frac{d^n f(x)}{dx^n} = \lim_{\epsilon \to 0+} \int_{-\infty}^{+\infty} E^n(x - y, \epsilon) f(y) dy \tag{2.26}$$

for any $n \geq 0$ for which the integral converges. So it is possible to identify

$$\begin{aligned} D^n(w) &= \frac{d^n}{dw^n}\delta(w) \\ \frac{d^n f(x)}{dx^n} &= \int_{-\infty}^{+\infty} D^n(x-y) f(y) dy, \end{aligned} \tag{2.27}$$

i.e. action of the operator $\mathbf{D}^n$ corresponds to *n-fold* derivative.

a2) α is a real, non integer number

Taking into account the cut orientations (2.19) calculation of limits (2.17) gives

$$\begin{aligned} D_-^\alpha(w) &= 0 && w > 0 \\ D_+^\alpha(w) &= 0 && w < 0 \\ D_-^\alpha(w) &= -\frac{\Gamma(\alpha+1)\sin([\alpha+1]\pi)}{\pi w^{\alpha+1}} && w < 0 \\ D_+^\alpha(w) &= +\frac{\Gamma(\alpha+1)\sin([\alpha+1]\pi)}{\pi w^{\alpha+1}} && w > 0. \end{aligned} \tag{2.28}$$

The first two equations consist with Eqs. (2.9), therefore again confirm correct correspondence of cuts in (2.19). After inserting of (2.28) into Eq. (2.4) we get

$$f_\pm^\alpha(x) = -\frac{\Gamma(\alpha+1)\sin([\alpha+1]\pi)}{\pi}\int_x^{\mp\infty}\frac{f(y)dy}{(x-y)^{\alpha+1}}. \tag{2.29}$$

For $y = x$ and $\alpha \geq 0$ the integral has a singularity. The method for regularization of this integral will become apparent in the next.

b) The action of operator $\mathbf{D}_\pm^\alpha$ as a difference of two integrals

After inserting of (2.18) to (2.4) we get

$$f_\pm^\alpha(x) = \lim_{\epsilon\to 0+}\left[\frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi}\left(\int_{-\infty}^{+\infty}\frac{f(y)dy}{(x-y+i\epsilon)^{\alpha+1}} - \int_{-\infty}^{+\infty}\frac{f(y)dy}{(x-y-i\epsilon)^{\alpha+1}}\right)\right]. \tag{2.30}$$

We assume for the present the function $f(y)$ is analytic on the whole real axis. Last equation can be rewritten

$$f_\pm^\alpha(x) = \lim_{\epsilon\to 0+}\left[\frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi}\left(\int_{-\infty-i\epsilon}^{+\infty-i\epsilon}\frac{f(z)dz}{(x-z)^{\alpha+1}} - \int_{-\infty+i\epsilon}^{+\infty+i\epsilon}\frac{f(z)dz}{(x-z)^{\alpha+1}}\right)\right]. \tag{2.31}$$

Again let us separate two cases:

b1) $\alpha = n \geq 0$ is an integer number

In Eq. (2.31) we can link up both integration paths in $z = \pm\infty$ and write

$$f^n(x) = \lim_{\epsilon\to 0+}\frac{(-1)^{n+1}\Gamma(n+1)}{2i\pi}\int_C\frac{f(z)dz}{(x-z)^{n+1}} = \frac{d^n f}{dx^n} \tag{2.32}$$

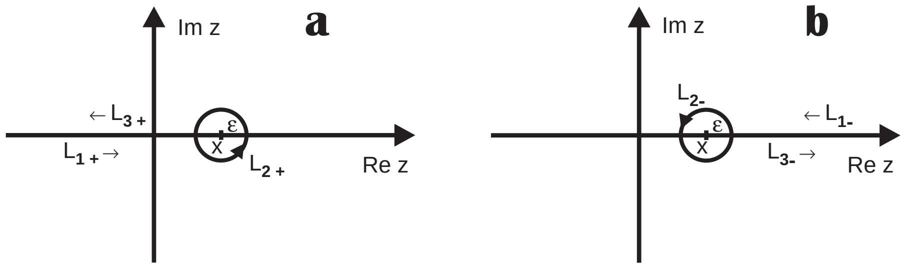

Figure 2: Integration paths in Eq. (2.33): $a)C_+$, $b)C_-$.

where C is any closed curve enclosing the singular point. Therefore for $n \geq 0$ the operator $\mathbf{D}^n$ can be again identified with ordinary *n-fold* derivative.

*b*2) *α is a real, non integer number*

Similarly, as in the case *b*1) we get the integrals

$$f^\alpha_\pm(x) = \lim_{\epsilon \to 0+} \frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi} \int_{C_\pm} \frac{f(z)dz}{(x-z)^{\alpha+1}} \tag{2.33}$$

where the paths $C_\pm$ can pass about the corresponding cut as shown in Fig.2. The integrals converge provided that

$$\lim_{z \to \pm\infty} \frac{f(z)}{z^\alpha} = 0. \tag{2.34}$$

The integration paths can be decomposed into three parts $L_{1\pm}$, $L_{2\pm}$, $L_{3\pm}$ and after evaluation of the corresponding integrals one gets

$$f^\alpha_\pm(x) = -\lim_{\epsilon \to 0+} \frac{\Gamma(\alpha+1)\sin([\alpha+1]\pi)}{\pi} \left(\int_{x\mp\epsilon}^{\mp\infty} \frac{f(z)dz}{(x-z)^{\alpha+1}} + \frac{(\pm 1)^\alpha f(x)}{\alpha\epsilon^\alpha} \right), \tag{2.35}$$

or using the relation (A.10) from Appendix,

$$f^\alpha_\pm(x) = -\lim_{\epsilon \to 0+} \frac{1}{\Gamma(-\alpha)} \left(\int_{x\mp\epsilon}^{\mp\infty} \frac{f(z)dz}{(x-z)^{\alpha+1}} + \frac{(\pm 1)^\alpha f(x)}{\alpha\epsilon^\alpha} \right). \tag{2.36}$$

For $\alpha < 0$ the integrals are finite and the second terms vanish. In this case Eq. (2.35) is identical with Eq. (2.29). On the other hand we have assumed the initial integral (2.33) is finite, therefore the sum of both terms in (2.35) is also finite even for any $\alpha > 0$. In this sense by addition of the second term the integral in (2.29) can be regularized. Note, that in contradistinction to (2.18) the relation (2.36) is well defined also for $\alpha = -1, -2, -3, ...$ but $\alpha \neq 0, 1, 2, 3, ...$ The last equation for α *negative integer*

$$f^{-n}_\pm(x) = \frac{1}{\Gamma(n)} \int_{\mp\infty}^{x} (x-z)^{n-1} f(z)dz \tag{2.37}$$

is a special case of the *n-fold* integral formula (1.1).

Now, let us assume $0 < \alpha < 1$ and calculate integral (2.33) by parts. Obviously

$$\int_{C_\pm} \frac{f(z)dz}{(x-z)^{\alpha+1}} = -\frac{1}{\alpha}\int_{C_\pm} \frac{f'(z)dz}{(x-z)^{\alpha}} + \left[\frac{f(z)}{(x-z)^{\alpha}}\right]_{\mp\infty\mp i0}^{\mp\infty\pm i0} \tag{2.38}$$

and integral on right side is finite. Any α can be written as the sum of integer and fractional part

$$\alpha = n + \Delta\alpha, \qquad n = [\alpha], \qquad 0 \le \Delta\alpha < 1. \tag{2.39}$$

For $n \ge 0$ we can repeat integration by parts $n+1$ times and if the function f meets requirements

$$\lim_{z\to\pm\infty} \frac{f^{p}(z)}{(x-z)^{\alpha-p}} = 0 \qquad \text{for } \ p = 0, 1, ...n, \tag{2.40}$$

then instead of (2.35) we get

$$f_\pm^\alpha(x) = \lim_{\epsilon\to 0+} \frac{(-1)^{\Delta\alpha}\Gamma(\Delta\alpha)}{2i\pi} \int_{C_\pm} \frac{f^{n+1}(z)dz}{(x-z)^{\Delta\alpha}} =$$

$$= -\frac{\Gamma(\Delta\alpha)\sin(\Delta\alpha\pi)}{\pi} \int_x^{\mp\infty} \frac{f^{n+1}(z)dz}{(x-z)^{\Delta\alpha}}. \tag{2.41}$$

Let us note that integrals (2.33) can be modified also in other way. We assume that all integrals

$$I_\pm(\gamma, x) = \int_{C_\pm} \frac{f(z)dz}{(x-z)^{\gamma}}, \qquad \gamma = \Delta\alpha, \ \Delta\alpha+1, ...\alpha+1 \tag{2.42}$$

converge, then recurrent relation holds

$$\frac{d}{dx} I_\pm(\gamma, x) = -\gamma I_\pm(\gamma+1, x). \tag{2.43}$$

Application of this relation for integral (2.33) and $\alpha = n + \Delta\alpha$ gives

$$f_\pm^\alpha(x) = \lim_{\epsilon\to 0+} \frac{(-1)^{\Delta\alpha}\Gamma(\Delta\alpha)}{2i\pi} \left(\frac{d}{dx}\right)^{n+1} \int_{C_\pm} \frac{f(z)dz}{(x-z)^{\Delta\alpha}} =$$

$$= -\frac{\Gamma(\Delta\alpha)\sin(\Delta\alpha\pi)}{\pi} \left(\frac{d}{dx}\right)^{n+1} \int_x^{\mp\infty} \frac{f(z)dz}{(x-z)^{\Delta\alpha}}. \tag{2.44}$$

Actually in Eq. (2.41) we apply operation $\mathbf{D}_\pm^{\Delta\alpha-1}$ on *n+1 - fold* derivative of the function f whereas in Eq. (2.44) the both operations are interchanged. Using relation (A.10) obviously we can modify both equations:

$$f_\pm^\alpha(x) = \frac{1}{\Gamma(1-\Delta\alpha)} \int_{\mp\infty}^{x} \frac{f^{n+1}(z)dz}{(x-z)^{\Delta\alpha}} = \frac{1}{\Gamma(1-\Delta\alpha)} \left(\frac{d}{dx}\right)^{n+1} \int_{\mp\infty}^{x} \frac{f(z)dz}{(x-z)^{\Delta\alpha}}. \tag{2.45}$$

So we have shown the operator $\mathbf{D}^\alpha$ with the kernel (2.18) can be considered a continuous interpolation of the ordinary *n-fold* derivative (integral) of the functions analytic on real axis and fulfilling the condition (2.34).

3 Composition of fractional derivatives

In this section we shall investigate how the composition of the operators $\mathbf{D}^\alpha$ is realized in the representation given by Eq. (2.18). Therefore we shall deal with the integrals

$$I = \int_{-\infty}^{+\infty} D^\alpha(x-y)D^\beta(y-z)dy. \tag{3.1}$$

Let us denote

$$\begin{aligned} h^\bullet(\gamma, w) &= \frac{1}{(w+i\tau)^\gamma} \\ h_\bullet(\gamma, w) &= -\frac{1}{(w-i\tau)^\gamma} \end{aligned} \qquad \tau > 0 \tag{3.2}$$

and

$$\begin{aligned} I_1 &= \textstyle\int_{-\infty}^{+\infty} h^\bullet(\alpha+1, x-y)h^\bullet(\beta+1, y-z)dy \\ I_2 &= \textstyle\int_{-\infty}^{+\infty} h_\bullet(\alpha+1, x-y)h_\bullet(\beta+1, y-z)dy \\ I_3 &= \textstyle\int_{-\infty}^{+\infty} h_\bullet(\alpha+1, x-y)h^\bullet(\beta+1, y-z)dy \\ I_4 &= \textstyle\int_{-\infty}^{+\infty} h^\bullet(\alpha+1, x-y)h_\bullet(\beta+1, y-z)dy, \end{aligned} \tag{3.3}$$

then

$$I = \frac{(-1)^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4\pi^2}(I_1+I_2+I_3+I_4). \tag{3.4}$$

Now let us calculate the more general integral

$$J = \int_K \frac{dz}{(z_2-z)^{\alpha+1}(z-z_1)^{\beta+1}}, \tag{3.5}$$

where K is arbitrary line in complex plane, z_1, z_2 any two (diverse) points and let $\alpha+\beta > -1$. We also assume the cuts of the function in the integral do not intersect the line K. Then there are two possibilities:

a) K is not passing between the points z_1, z_2, see Fig.3*a*. Then obviously $J = 0$, since the line K can be closed in infinity by the arc in half plane which do not contain singularities z_1, z_2.

b) K is passing between the points z_1, z_2, see Fig.3*b*. K' denotes line crossing the segment $\langle z_1, z_2\rangle$ perpendicularly at its center. If we assume, that the cuts do not intersect any of both lines K, K', then in the integral (3.5) path K can be substituted by K'. Further, if we denote

$$z_0 = (z_1+z_2)/2, \qquad r\exp(i\varphi) = (z_2-z_1)/2 \tag{3.6}$$

and substitute $z = z_0 + it\exp(i\varphi)$, then

$$J = \frac{i}{\exp[i\varphi(\alpha+\beta+1)]}\int_{-\infty}^{+\infty} \frac{dt}{(r-it)^{\alpha+1}(r+it)^{\beta+1}}. \tag{3.7}$$

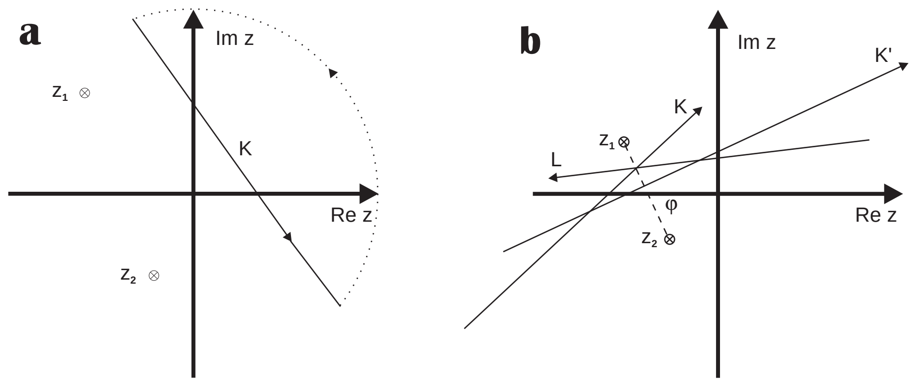

Figure 3: Integration paths in Eq. (3.5): *a*)case when the integral vanishes, *b*)case leading to the result (3.8).

The last integral can be found in tables (see e.g. [7],p.301), using (3.6) we obtain

$$J = \frac{2i\pi}{(z_2 - z_1)^{\alpha+\beta+1}} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}. \tag{3.8}$$

Let us note that the opposite orientation of the line K (i.e. point z_2 on left side in respect to the direction of K, as the line L in Fig.3b) should give result (3.8) with opposite sign. Obviously the integrals I_3, I_4 vanish and for I_1, I_2 we get

$$I_1 = \frac{-2i\pi}{(x - z + 2i\tau)^{\alpha+\beta+1}} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \tag{3.9}$$

$$I_2 = \frac{+2i\pi}{(x - z - 2i\tau)^{\alpha+\beta+1}} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \tag{3.10}$$

and inserting into (3.4) gives

$$I = \frac{(-1)^{\alpha+\beta+1}\Gamma(\alpha+\beta+1)}{2i\pi} \left(\frac{1}{(x - z + 2i\tau)^{\alpha+\beta+1}} - \frac{1}{(x - z - 2i\tau)^{\alpha+\beta+1}} \right). \tag{3.11}$$

For $\tau \to 0$ this expression corresponds to $D^{\alpha+\beta}(x - z)$ in (2.18).

This result is formally correct, nevertheless its drawback is that it does not reflect the correspondence of cut orientations in initial expressions (3.1) and the final (3.11). More rigorous discussion about the cuts we postpone to Appendix, here give only result: the following composition relation holds for operators (2.18) with equally oriented cuts

$$D_{\pm}^{\alpha+\beta}(x - z) = \int_{-\infty}^{+\infty} D_{\pm}^{\alpha}(x - y) D_{\pm}^{\beta}(y - z) dy \qquad \alpha, \beta \neq -1, -2, -3...; \quad \alpha + \beta > -1. \tag{3.12}$$

Let us note, that validity of the relation (3.12) can be verified in the initial representation (2.7) as well. Further, considering $D^\alpha_\pm$ as a generalized function, then the condition $\alpha+\beta>-1$ can be omitted and the composition relation has the form

$$\int_{-\infty}^{+\infty} D^{\alpha+\beta}_\pm(x-\xi)f(\xi)d\xi = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} D^\alpha_\pm(x-y)D^\beta_\pm(y-\xi)f(\xi)d\xi dy, \quad \alpha,\beta \neq -1,-2,.. \tag{3.13}$$

for all functions analytic on the real axis and fulfilling

$$\lim_{z\to\pm\infty}\frac{f(z)}{z^{\alpha+\beta}}=0. \tag{3.14}$$

Eq. (3.13) follows from (3.12) and relation $\frac{d}{dw}D^{\gamma-1}_\pm(w)=D^\gamma_\pm(w)$. Repetitional integration by parts on both sides can reduce sum $\alpha+\beta$ by any natural number, so in this way validity of (3.13) is proved.

All our previous considerations concerned acting of the operator $\mathbf{D}^\alpha$ on real axis. In the next we shall try to enlarge them on whole complex plane.

4 Fractional derivative in complex plane

First, let us illustrate the notion fractional derivative introduced in previous part on a particular example. Take the function

$$f(x)=\frac{1}{1+x^2}=-\frac{1}{2i}\left(\frac{1}{x+i}-\frac{1}{x-i}\right), \tag{4.1}$$

for which (2.33) gives

$$f^\alpha_\pm(x)=\lim_{\epsilon\to 0+}\frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi}\int_{C_\pm}\frac{dz}{(x-z)^{\alpha+1}(1+z^2)}. \tag{4.2}$$

Obviously for $\alpha>-2$, the integration paths in Fig.2 closed by arcs in infinity as shown in Fig.4, are possible and

$$f^\alpha_\pm(x)=\frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i}\left(\frac{1}{(x+i)^{\alpha+1}}-\frac{1}{(x-i)^{\alpha+1}}\right). \tag{4.3}$$

Even though the result formally does not depend on given subscript ($+$ or $-$), it is necessary to take into account different cut orientation for f^α_+, f^α_-. In accordance with the phase convention (2.6), we take

$$\left.\begin{aligned}(x+i)^{\alpha+1} &= R^{\alpha+1}\exp[i\varphi(\alpha+1)]\\ (x-i)^{\alpha+1} &= R^{\alpha+1}\exp[i(2\pi-\varphi)(\alpha+1)]\end{aligned}\right\}\quad \text{for } f^\alpha_+$$
$$\left.\begin{aligned}(x+i)^{\alpha+1} &= R^{\alpha+1}\exp[i\varphi(\alpha+1)]\\ (x-i)^{\alpha+1} &= R^{\alpha+1}\exp[-i\varphi(\alpha+1)]\end{aligned}\right\}\quad \text{for } f^\alpha_-, \tag{4.4}$$

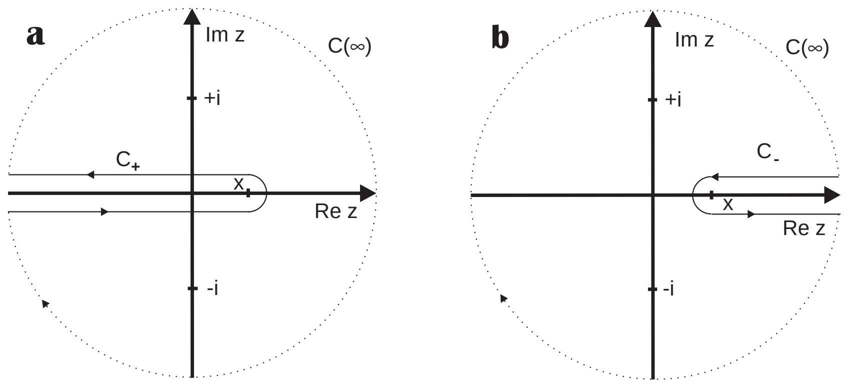

Figure 4: Integration paths in Eq. (4.2) closed in infinity: $a)C_+,\ b)C_-$.

where

$$R = \sqrt{1+x^2}, \qquad \varphi = \frac{\pi}{2} - \arcsin\frac{x}{R}. \tag{4.5}$$

After inserting into (4.3) and a simple rearrangement, we obtain

$$f^{\alpha}_{\pm}(x) = \left(\frac{\pm 1}{\sqrt{1+x^2}}\right)^{\alpha+1} \Gamma(\alpha+1)\sin\left[\left(\arcsin\frac{x}{\sqrt{1+x^2}} \pm \frac{\pi}{2}\right)(\alpha+1)\right]. \tag{4.6}$$

Remark: From the last relation it can be easily shown that e.g.

a) for α integer, $\alpha = n > -1$, it holds $f^n_+(x) = f^n_-(x) \equiv f^n(x)$, where f^n is ordinary *n-fold* derivative

b) $f^{\alpha}_-(x) = (-1)^{\alpha} f^{\alpha}_+(-x)$

c) for $\alpha \to -1$ we get a primitive function to f : $\quad f^{\alpha}_{\pm}(x) \to \arctan(x) \pm \pi/2$. The same results can be obtained also from (2.45) for $n = -1, \quad \Delta\alpha = 0$.

Using formula (2.33) we shall try to generalize the operator of fractional derivative on real axis to the whole complex plane. For this operator we shall demand again:

$$\mathbf{D}^{\alpha} f(z) = \frac{d^n f}{dz^n} \qquad \text{for } \alpha = n \geq 0, \tag{4.7}$$

$$\mathbf{D}^{\alpha} \circ \mathbf{D}^{\beta} = \mathbf{D}^{\alpha+\beta}. \tag{4.8}$$

For making the generalization more transparent, we shall do it in several steps.

1) Let us go back to Fig.4 and put a question what will change in results (4.3)$-$(4.6), if the cut (and integration path correspondingly) would be oriented otherwise than along real axis, but e.g. as shown in Fig.5. Obviously for Eq. (4.3) nothing will change, but the form of Eqs. (4.4),(4.6) will depend on the mutual position of the cut and both poles.

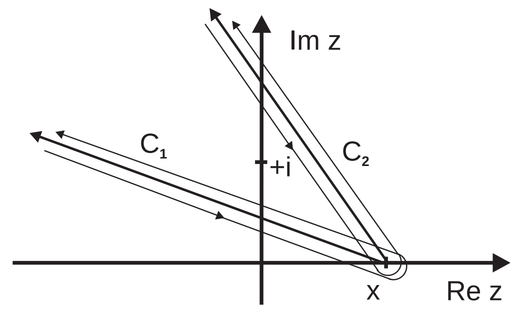

Figure 5: Tentative modification of integration paths in Eq. (4.2) and Fig.4.

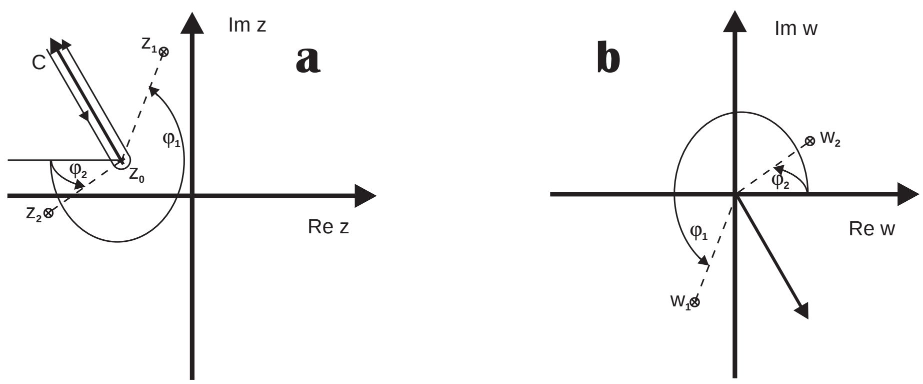

Figure 6: a)Integration paths in Eq. (2.33) for the function (4.9). φ_1, φ_2 are phases corresponding to terms $w_k^{\alpha+1} = (z_0 - z_k)^{\alpha+1}$ in the result of integration (4.11). b) The same phases represented for variable w.

Because we have accepted phase convention (2.6) only for cuts on real axis, it is now necessary to make a more general consideration to determine the phases of both terms in (4.3). Instead of (4.1) let us take a complex function

$$f(z) = \frac{a_1}{z - z_1} + \frac{a_2}{z - z_2} \tag{4.9}$$

and integration paths C displayed in Fig.6a. The corresponding integral will be

$$f^\alpha(z_0) = -(-1)^{\alpha+1}\Gamma(\alpha+1)\left(\frac{a_1}{(z_0 - z_1)^{\alpha+1}} + \frac{a_2}{(z_0 - z_2)^{\alpha+1}}\right). \tag{4.10}$$

The path C passes about the cut of function $1/w^{\alpha+1} = 1/(z_0 - z)^{\alpha+1}$ in variable z, i.e. cut of the function $1/w^{\alpha+1}$ is oriented in opposite direction, see Fig.6b. The phases of $w_1^{\alpha+1} = (z_0 - z_1)^{\alpha+1}$ and $w_2^{\alpha+1} = (z_0 - z_2)^{\alpha+1}$ must be fixed in respect to this cut. Fig.6b prompts the following rule.

Rule 1: *Phase of complex variable w is given by the angle φ of arc leading from positive real half axis and measured in positive direction (against clockwise sense) to the point w and if the arc intersects the cut, φ is reduced by 2π.*
This rule can be applied also directly for situation in Fig.6a for fixing phase of $(z_0 - z)^{\alpha+1}$. The only modification is that φ is measured from half line $(z_0, z_0 - \infty)$. Therefore angles φ_1, φ_2 in Fig.6 fix phases in (4.10)

$$f^{\alpha}(z_0) = (-1)^{\alpha}\Gamma(\alpha+1)\left(\frac{a_1 \exp(-i\varphi_1[\alpha+1])}{\mid (z_0 - z_1)^{\alpha+1} \mid} + \frac{a_2 \exp(-i\varphi_2[\alpha+1])}{\mid (z_0 - z_2)^{\alpha+1} \mid}\right). \tag{4.11}$$

Now the introduced rule can be applied also for integration on paths $C_1(C_2)$ in Fig.5. Doing this, we shall get the same result as in the case of integration on paths $C_+(C_-)$ in Fig.4. Therefore depending on the position of chosen derivative cut in respect to the both poles, the function (4.1) has (up to factor (2.8)) in a given point x two different values of fractional derivative given by Eq. (4.6).

2)We shall now generalize the prescription for calculating of fractional derivative of function having the form (4.9) for functions having a finite number of poles

$$h(z) = \sum_{k=1}^{N} \frac{a_k}{(z - z_k)^{n_k+1}}, \qquad n_k \geq 0. \tag{4.12}$$

Let the derivative cut be given by half line $L \equiv [z_0,\ z_0 + \exp(i\theta)\infty]$ which does not go through any of poles z_k. Then

$$h^{\alpha}(z_0) = \lim_{\epsilon \to 0+} \frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi} \sum_{k=1}^{N} \int_{C(L)} \frac{a_k dz}{(z_0 - z)^{\alpha+1}(z - z_k)^{n_k+1}}$$

$$= (-1)^{\alpha} \sum_{k=1}^{N} \frac{\Gamma(\alpha + n_k + 1)}{\Gamma(n_k + 1)} \frac{a_k \exp(-i\varphi_k[\alpha+1])}{\mid z_0 - z_k \mid^{\alpha+1} (z_0 - z_k)^{n_k}}, \tag{4.13}$$

integration path $C(L)$ is shown in Fig.7. Angles φ_k are calculated using the **Rule 1**, therefore it is obvious, that the function $h^{\alpha}(z_0)$ in (4.13) can have, according to the cut orientation, as much values, as much different poles z_k the function (4.12) has.

3) Now let us consider functions (4.12),(4.13) in the case, when the cut is not a line, but some general curve connecting the points z_0 and $\exp(i\theta)\infty$. Then the result (4.13) will be formally the same, only the prescription for fixing of angles φ_k must be modified. General cases are illustrated in Fig.8a,b, where it is apparent that the arc, which 'measures' angle, may intersect the cut several times. The consistent assigning of angles φ_k corresponding to points z_k in Eq. (4.13) can be ensured by the following prescription.
Rule 2: *Let us choose on the half line $(z_0, z_0 - \infty)$ any reference point z_R. Angle φ_k is given by angle $z_R z_0 z_k$ measured in positive sense and then for each intersection with the cut is corrected as follows. Superpose palm of right or left hand at an intersection in*

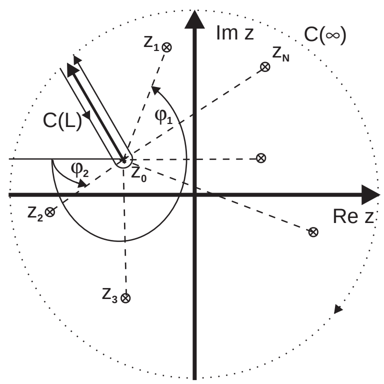

Figure 7: Integration paths in Eq. (2.33) with the function (4.12) and phases φ_k appearing in the result of integration (4.13).

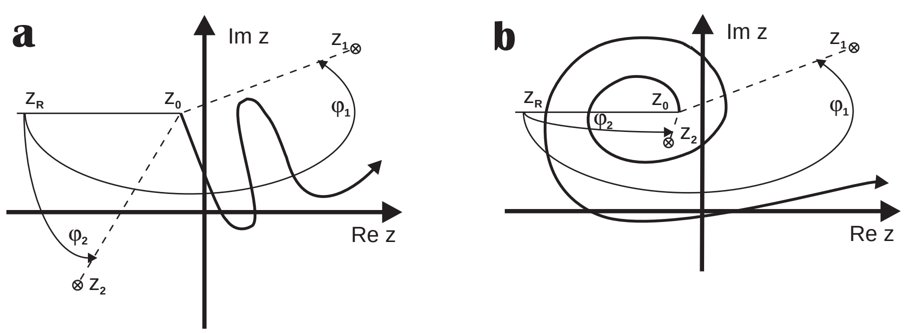

Figure 8: a), b) Examples of the curvilinear cuts generating integration paths for Eq. (2.33) with the function (4.12). Other symbols are defined in **Rule 2**.

such a way, that fingers lead from z_R towards z_k and thumb leads in direction of the cut from branching point z_0. If this condition is met by right (left) hand, φ_k will be enhanced (reduced) by 2π.

Remark: It is substantial for all z_k to choose one common reference point z_R. A shift of this point results at most in only equal shifts of all angles $\varphi_k \to \varphi_k + 2n\pi$, which do not change Eq. (4.13).

Therefore for any curvilinear cut the relation (4.13) can be written

$$h^\alpha(z_0) = (-1)^\alpha \sum_{k=1}^{N} \frac{\Gamma(\alpha+n_k+1)}{\Gamma(n_k+1)} \frac{a_k \exp(-i[\varphi_k + 2m_k\pi][\alpha+1])}{\mid z_0 - z_k \mid^{\alpha+1} (z_0 - z_k)^{n_k}}. \tag{4.14}$$

The set of integer numbers (or more exactly their differences) characterizes the way how the curvilinear cut passes among the poles z_k. If we accept curvilinear cuts for derivative of the function (4.1), then using (4.14) one can obtain

$$f^\alpha(x) = \frac{(-1)^{k(\alpha+1)}\Gamma(\alpha+1)}{\left(\sqrt{1+x^2}\right)^{\alpha+1}} \sin\left[\left(\arcsin\frac{x}{\sqrt{1+x^2}} + (2k+1)\frac{\pi}{2}\right)(\alpha+1)\right], \tag{4.15}$$

where $k = m_2 - m_1$. Let us note that

$$\lim_{\alpha\to-1} f^\alpha(x) = \arctan(x) + (2k+1)\frac{\pi}{2}, \tag{4.16}$$

i.e. we obtain the infinite (but countable) set of primitive functions for the function (4.1).

4) So far we have considered only analytic functions of the form (4.12), for which the corresponding integral on whole circle $C(\infty)$ vanishes. Now we are going to consider the general case, when this integral vanishes only on a part of the circle. But first, let us go back to the operator (2.18) acting on analytic functions on real axis (or the part of this axis) and try to generalize it for analytic functions on a curve in complex plane connecting a pair of points on the circle $C(\infty)$, see Fig.9*a*. Let this curve be given as a continuous complex function of real parameter $t \in (-\infty, +\infty)$

$$z = x(t) + iy(t) \equiv \psi(t), \qquad \psi(-\infty) = \infty_1, \qquad \psi(+\infty) = \infty_2, \tag{4.17}$$

then its derivative

$$\psi'(t) = \frac{dx}{dt} + i\frac{dy}{dt} \tag{4.18}$$

determines in complex plane the vector tangent to curve ψ and oriented in direction of increasing t. The function

$$\nu(t) = \frac{\psi'(t)}{\sqrt{\psi'(t)\cdot\psi^{*\prime}(t)}} = \exp[i\omega(t)] \tag{4.19}$$

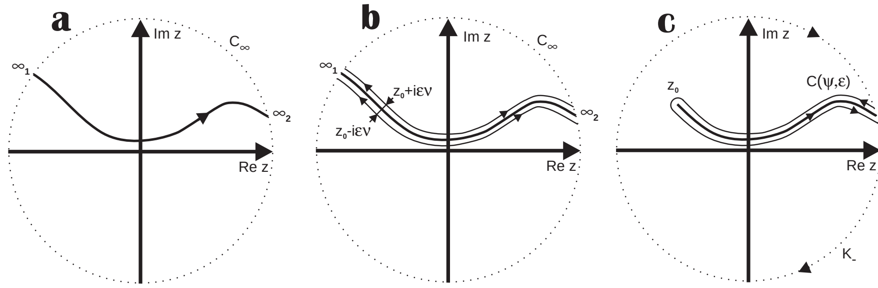

Figure 9: a)Integration paths for operator (4.20) in complex plane. b) Corresponding curvilinear cuts in Eq. (4.20). c) Corresponding integration path in Eq. (4.23).

represents this vector in its normalized value ($\omega(t)$ is phase of this vector) and the function $i\nu(t)$ normalized vector perpendicular to ψ and oriented left in respect to the course of ψ.

Now let us define the function of the complex variable $z \in \psi$ with complex parameters $z_0 \in \psi$ and $\alpha \neq -1, -2, -3, ...$

$$D^\alpha_\psi(z_0 - z) = \lim_{\epsilon \to 0+} \frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi} \left(\frac{1}{(z_0 - z + i\epsilon\nu)^{\alpha+1}} - \frac{1}{(z_0 - z - i\epsilon\nu)^{\alpha+1}} \right) \quad (4.20)$$

having the curvilinear cuts (for $\alpha \neq 0, 1, 2, ..$) corresponding to the both terms coming out the points $z_0 \pm i\epsilon\nu$ and going jointly along the curve ψ to points ∞_1 or ∞_2, see Fig.9b. So, in contradistinction to linear cuts the form of the cut of $w^{\alpha+1} = (z_0 - z)^{\alpha+1}$ does depend on position of point z_0 on ψ.

Next, using this function we define the operator

$$\mathbf{D}^\alpha_\psi\ f = f^\alpha_{\psi\pm}(z_0) = \int_\psi D^\alpha_\psi(z_0 - z) f(z) dz = \int_{-\infty}^{+\infty} D^\alpha_\psi(\psi(t_0) - \psi(t)) f(\psi(t))\psi'(t) dt \quad (4.21)$$

acting on the functions analytic on the curve ψ for which the integral converges.
Remark: Let us note the integral (4.21) depends on choice of cut end-point (∞_1 or ∞_2) but does not depend on in which direction ($\infty_1 \to \infty_2$ or $\infty_1 \leftarrow \infty_2$) integration is done. That is a consequence of the fact that change of integration direction $\psi(t) \to \psi(-t),\ dz \to -dz$ implies the change $\nu(t) \to -\nu(t)$, which implies the change $D^\alpha_\psi(z_0 - z) \to -D^\alpha_\psi(z_0 - z)$, therefore the integral does not change.

If we calculate (4.21) as the difference

$$\mathbf{D}^\alpha_\psi\ f = \lim_{\epsilon \to 0+} \frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi} \left(\int_\psi \frac{f(z)dz}{(z_0 - z + i\epsilon\nu)^{\alpha+1}} - \int_\psi \frac{f(z)dz}{(z_0 - z - i\epsilon\nu)^{\alpha+1}} \right), \quad (4.22)$$

then this difference can be expressed as

$$\int_\psi D^\alpha_\psi(z_0 - z) f(z) dz = \lim_{\epsilon \to 0+} \frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi} \int_{C(\psi,\epsilon)} \frac{f(z)dz}{(z_0 - z)^{\alpha+1}}, \quad (4.23)$$

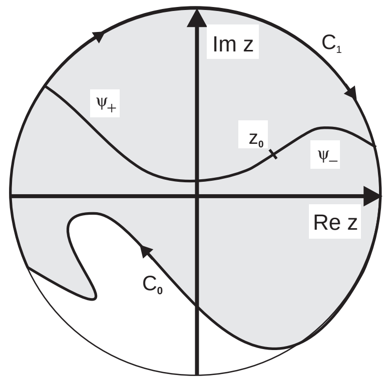

Figure 10: Domain G (grey area) and the curves defined in assumptions of the **Theorem**.

where the path in 'distance' ϵ passes about the cut coming out branching point z_0 on the curve ψ to infinity, see Fig.9c.

To label the integration cuts ending either at ∞_1 or ∞_2 we can accept the following convention. Let $\infty_1 = \exp(i\theta_1)\infty$ and $\infty_2 = \exp(i\theta_2)\infty$, then for

$$\begin{array}{lll} \cos\theta_1 \neq \cos\theta_2 & \left\{ \begin{array}{c} \cos\theta_1 < \cos\theta_2 \\ \cos\theta_1 > \cos\theta_2 \end{array} \right\} \quad \text{we define} & \left\{ \begin{array}{ll} \psi_+ = (z_0, \infty_1), & \psi_- = (z_0, \infty_2) \\ \psi_+ = (z_0, \infty_2), & \psi_- = (z_0, \infty_1) \end{array} \right\} \\ \cos\theta_1 = \cos\theta_2 & \left\{ \begin{array}{c} \sin\theta_1 < \sin\theta_2 \\ \sin\theta_1 > \sin\theta_2 \end{array} \right\} \quad \text{we define} & \left\{ \begin{array}{ll} \psi_+ = (z_0, \infty_1), & \psi_- = (z_0, \infty_2) \\ \psi_+ = (z_0, \infty_2), & \psi_- = (z_0, \infty_1) \end{array} \right\} \end{array} \tag{4.24}$$

and correspondingly we index related symbols $\mathbf{D}^{\alpha}_{\psi\pm},\ D^{\alpha}_{\psi\pm},\ \ f^{\alpha}_{\psi\pm},\ C_{\pm}(\psi,\epsilon)$.

Let us note the definition of the operator $\mathbf{D}^{\alpha}_{\psi}$ by Eqs. (4.20), (4.21) ensure that the both curves $C_{\pm}(\psi,\epsilon)$ are oriented in such a way that after their closing by the circle $C(\infty)$ see Fig.9c, there arises closed curve having *always* clockwise orientation. We denote these curves as $K_{\pm}(\psi,\epsilon)$.

Now, after all these preparing steps we can come to the formulation and proof of the following theorem.

Theorem

Assumptions:

i) G is the domain in complex plane containing the part (or parts) $C_1 \equiv G \cap C(\infty)$ of the circle $C(\infty)$. The curve C_1 forms one part, the curve C_0 is the second part and closed curve $C_G \equiv C_0 \cup C_1$ constitutes the complete domain boundary (Fig. 10).

ii) ψ is a curve defined in (4.17) and $\psi \subset G,\ \psi \cap C_0 = \emptyset$.

iii) Function $f(z)$ is analytic in G except for the poles $z_k \notin \psi,\ z_k \notin C_G,\ k = 1...N$ and for a given $\alpha \equiv \alpha_1 + i\alpha_2 \neq -1, -2, -3...$ and any $z_1 \in C_1$ meets the condition

$$\lim_{z \to z_1} \frac{f(z)}{z^{\alpha_1}} = 0. \tag{4.25}$$

Statements:

1. Operation

$$\mathbf{D}^{\alpha}_{\psi\pm}f = f^{\alpha}_{\psi\pm}(z_0) = \int_{\psi} D^{\alpha}_{\psi\pm}(z_0 - z)f(z)dz, \qquad z_0 \in \psi \tag{4.26}$$

is a fractional derivative satisfying the conditions (4.7), (4.8). In particular that means the operation does not depend on ψ if $\alpha = n \geq 0$ is integer.

2. $f^{\alpha}_{\psi\pm}$ is given also equivalently by

$$f^{\alpha}_{\psi\pm}(z_0) = \lim_{\epsilon\to 0+} \frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi} \int_{C_{\pm}(\psi,\epsilon)} \frac{f(z)dz}{(z_0 - z)^{\alpha+1}}. \tag{4.27}$$

3. For $N = 0$ the value $f^{\alpha}_{\psi\pm}(z_0)$ does not depend on ψ, i.e. up to factor $(-1)^{\alpha}$ is uniquely defined.

4. For $N \geq 1$ and α non integer there remain cases:

4.1. α_1 is rational and $\alpha_2 = 0$, then $f^{\alpha}_{\psi\pm}(z_0)$ does depend on ψ, but the set of values is finite.

4.2. α_1 is irrational or $\alpha_2 \neq 0$, then $f^{\alpha}_{\psi\pm}(z_0)$ does depend on ψ as well and in general the set of values is infinite, but countable.

Proof:
We shall start from the statement 2. Obviously, its validity follows from Eqs. (4.22), (4.23).

Now let us consider the statement 4. If we express the function $f(z)$ as the sum $f(z) = g(z) + h(z)$, where $g(z)$ is analytic in G and $h(z)$ has the form (4.12), then obviously

$$f^{\alpha}_{\psi\pm}(z_0) = \lim_{\epsilon\to 0+} \frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi} \int_{C_{\pm}(\psi,\epsilon)} \frac{f(z)dz}{(z_0 - z)^{\alpha+1}}$$

$$= \lim_{\epsilon\to 0+} \frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi} \int_{K_{\pm}(\psi,\epsilon)} \frac{f(z)dz}{(z_0 - z)^{\alpha+1}} - \frac{(-1)^{\alpha+1}\Gamma(\alpha+1)}{2i\pi} \int_{C_0} \frac{f(z)dz}{(z_0 - z)^{\alpha+1}} \tag{4.28}$$

$$= (-1)^{\alpha} \left(\sum_{k=1}^{N} \frac{\Gamma(\alpha + n_k + 1)}{\Gamma(n_k + 1)} \frac{a_k \exp(-i[\varphi_k + 2m_k\pi][\alpha+1])}{\mid z_0 - z_k \mid^{\alpha+1} (z_0 - z_k)^{n_k}} + I(z_0) \right),$$

where we have denoted

$$I(z_0) = \frac{\Gamma(\alpha+1)}{2i\pi} \int_{C_0} \frac{f(z)dz}{(z_0 - z)^{\alpha+1}}. \tag{4.29}$$

The angles $\varphi_k + 2m_k\pi$ corresponding to the poles z_k are evaluated according to **Rule 2.** In the function $I(z_0)$ the orientation of the path C_0 is accordant with $K_{\pm}$ and phase of $(z_0 - z)^{\alpha+1}$ must be also according to the **Rule 2** related to the same reference point z_R.

Now, if $\alpha_1 = p/q$ (p, q are not commensurable) and $\alpha_2 = 0$, then each term in the last sum has the same value also for $m'_k = m_k + q$, i.e. depending on the form of the cut and the corresponding set $\{m_k, 1 \leq k \leq N\}$ the expression (4.28) has only *finite* number of values for some z_0. On the other hand for α irrational different sets $\{m_k\}$ give different values of the sum, therefore in general the number of values is *infinite.* For $\alpha_2 \neq 0$ multi-value factors in the sum (4.28) are expanded

$$1^\alpha = \exp(2i\pi m\alpha_1) \cdot \exp(-2\pi m\alpha_2) \tag{4.30}$$

i.e. $m\alpha_1, m\alpha_2$ determine phase and scale of individual terms in the sum. Obviously for a complex α the set of values $f^\alpha(z_0)$ depending on ψ is in general *infinite*, similarly as for α irrational. Therefore the statements 4.1, 4.2 are proved.

For $N = 0$ Eq. (4.28) is simplified:

$$f^\alpha_{\psi\pm}(z_0) = (-1)^\alpha I(z_0) \tag{4.31}$$

and trueness of statement 3. is evident.

Finally, let us consider the statement 1. Validity of the condition (4.7) follows from the statement 2. For α integer the path $C_\pm(\psi, \epsilon)$ can be closed around the pole z_0 (having *positive* orientation) and we get the Cauchy integral

$$\frac{n!}{2i\pi} \int_{C(z_0)} \frac{f(z)dz}{(z - z_0)^{n+1}} = f^n(z_0). \tag{4.32}$$

The condition (4.8) requires validity

$$\int_\psi D^{\alpha+\beta}_{\psi\pm}(x - \xi) f(\xi) d\xi = \int_\psi \int_\psi D^\alpha_{\psi\pm}(x - y) D^\beta_{\psi\pm}(y - \xi) f(\xi) d\xi dy. \tag{4.33}$$

This relation follows from (4.21) and (3.13) in which the substitutions $\xi \to \psi(\xi)$ and $y \to \psi(y)$ are applied. So the whole proof is completed.

5 Discussion

5.1 Some remarks regarding the theorem

Now, let us look into the statements of the **Theorem** more comprehensively. The function $I(z_0)$ in (4.28) can be expressed

$$I(z_0) = \frac{\Gamma(\alpha + 1)}{2i\pi} \int_{C_0} \frac{f(z)dz}{(z_0 - z)^{\alpha+1}} =$$

$$= \frac{\Gamma(\alpha + 1) \exp(-i2m\pi[\alpha + 1])}{2i\pi} \int_{C_0} \frac{f(z) \exp[-i\varphi(z)(\alpha + 1)]dz}{|(z_0 - z)^{\alpha+1}|}. \tag{5.1}$$

Since the cut ψ does not intersect the curve C_0 the phases of all points on C_0 are 'corrected' by the same factor standing ahead of the integral. Now it is obvious that for $\alpha = p/q$ the function (4.28) can have at most q^{N+1} different values, including multi-value factor $(-1)^\alpha$.

Actually, the function (4.28) can be written as

$$f^\alpha(z) = (-1)^\alpha \left(\sum_{k=1}^{N} \frac{\Gamma(\alpha+n_k+1)}{\Gamma(n_k+1)} \frac{a_k}{(z-z_k)^{n_k+\alpha+1}} + \frac{\Gamma(\alpha+1)}{2i\pi} \int_{C_0} \frac{f(\xi)d\xi}{(z-\xi)^{\alpha+1}} \right) \tag{5.2}$$

and considered a multi-value function with the value at point z depending on the choice of cut ψ in (4.26). At given point z there are equivalent any two cuts for which the region closed by them and the curve C_1 does not contain any pole z_k. Obviously, except for the points z_k the function $f^\alpha(z)$ is (as original function f) analytic in the domain G. Moreover, in the region of α in which $f^\alpha = \mathbf{D}^\alpha f$ exists, this function is apparently analytic also in respect to α.

Special case takes place when α is a negative integer. Then due to the singularity $\Gamma(-n)$ the kernel (4.20) loses the sense. Nevertheless, assuming in (4.20) initially $\alpha \neq -n$ one can proceed to representation (4.27), then making the limit $\epsilon \to 0$ (like Eq. (2.35)) gives

$$f^\alpha_{\psi\pm}(z_0) = -\lim_{\epsilon\to 0+} \frac{\Gamma(\alpha+1)\sin([\alpha+1]\pi)}{\pi} \left(\int_{\psi\pm} \frac{f(z+\nu_0\epsilon)dz}{(z_0-z-\nu_0\epsilon)^{\alpha+1}} + \frac{f(z_0)}{\alpha(-\nu_0\epsilon)^\alpha} \right), \tag{5.3}$$

where ν_0 is given by Eq. (4.19) and represents direction of the cut at z_0 (orientation is assumed from z_0 to infinity). This formula already makes sense for $\alpha = -n$ (if condition (4.25) holds). Using relation (A.10) gives

$$f^{-n}_{\psi\pm}(z_0) = -\frac{1}{(n-1)!} \int_{\psi\pm} (z_0-z)^{n-1} f(z)dz, \tag{5.4}$$

which is a modification of the known formula (1.1) for *n-fold* primitive function. The set of values $f^{-n}(z_0)$ depends on ψ as follows. If $f(z)$ has no poles in the domain G, then any two integration paths ψ_1, ψ_2 in (5.4) can be connected by a fragment of C_1, in this way there arises closed curve C and it holds

$$0 = -\frac{1}{(n-1)!} \int_C (z_0-z)^{n-1} f(z)dz = f^{-n}_{\psi_2}(z_0) - f^{-n}_{\psi_1}(z_0) \tag{5.5}$$

i.e. $f^{-n}(z_0)$ is determined uniquely. Now, let us suppose $f(z)$ has inside curve C just one pole, for $z \to z_p$

$$f(z) \to \frac{a_p}{(z-z_p)^{n_p+1}}. \tag{5.6}$$

Then obviously any curve C in the integral (5.5) can be always substituted by a couple of curves K_0, each of them is closed in the phase range $\langle 0, 2\pi\langle$ and having the pole z_p inside

(e.g. circles centered at z_p). Instead of (5.5) one get the difference

$$\Delta_p \equiv f_{\psi_2}^{-n}(z_0) - f_{\psi_1}^{-n}(z_0) = \frac{ma_p}{(n-1)!} \int_{K_0} \frac{(z_0 - z)^{n-1}}{(z - z_p)^{n_p+1}} dz, \tag{5.7}$$

where m is the integer depending on the shape of the curve C (m represents the number of 'twists' on the C). Using the Cauchy formula, the last equation gives

$$\Delta_p = \begin{cases} 0 & n_p \geq n \\ \dfrac{2i\pi m a_p}{(n - n_p - 1)! n_p!} (z_0 - z_p)^{n - n_p - 1} & n_p < n \end{cases} \tag{5.8}$$

For more poles the all corresponding terms (5.8) are simply added. For example integration constants in (4.16) representing the case $n = 1$, $n_p = 0$ fulfill (5.8). The composition relation for $\alpha = -n$, $\beta < 0$ and $\psi \equiv$*real axis* is proved in Appendix (Eq. (A.17)) and apparently can be transformed to any curvilinear path ψ.

Let us note, for α *negative integer* only the representation of $\mathbf{D}^\alpha$ given by Eq. (5.3) (or (5.4)) makes sense and conversely for α *non negative integer* only the representation given by Eq. (4.26) with kernel (4.20) (or equivalently by Eq. (4.27)) is well defined. For any α *complex but non integer* both representations are well defined. These two representations differ in the corresponding integration paths:

i) Eq. (4.27) - integration on some curve enveloping the cut, the path can be closed.

ii) Eq. (5.3) - integration on the cut itself, the path cannot be closed.

For α *integer* it is specific, that the cuts disappear.

Perhaps most restrictive assumption in the **Theorem** is the condition (4.25). The question, if one can in some consistent way apply $\mathbf{D}^\alpha$ to the functions not obeying this condition in any part of $C(\infty)$ (and simultaneously having the ordinary derivatives - primitive functions) requires a further study. Obviously one possible way is to consider such functions as generalized functions as well.

5.2 Concluding remarks

Have we said something new? First let us show what is not new. Apparently:

a) The content of Eq. (2.45) is almost identical with Liouville definition of right and left -handed fractional derivatives in [8], p.95. The only distinction is in phase of f_-^α since in the cited definition for the real functions the real value of the derivatives is ad hoc postulated.

b) Eq. (4.27) is the Cauchy type integral which ordinarily serves as one of possible starting points for the definition of the fractional derivative in complex plane, see [8],p.415. In our approach the integration path is uniquely defined by chosen cut.

On the other hand, the new seems be the following:

1) The general form of the kernel (4.20) from which the both above mentioned formulae follow.

2) The construction based on the integration paths enveloping curvilinear cuts, which in the result allows to identify fractional derivative - integral with the multi-valued function and to determine how the number of values depends on the derivative order type and the number of poles which the given function has in the considered region.

Appendix: The correspondence of cuts in the composition relation
Similarly as in (3.2) we define

$$\begin{aligned} h^{\pm}(\gamma,w) &= \frac{1}{(w+i\tau)^{\gamma}} \\ h_{\pm}(\gamma,w) &= -\frac{1}{(w-i\tau)^{\gamma}} \end{aligned} \qquad \tau>0, \tag{A.1}$$

where indices $\pm$ denote the cut orientations $(0,\pm\infty)$. Let us consider the integrals

$$I=\lim_{\tau\to 0+}\int_{-\infty}^{+\infty} h(a,x-y)h(b,y-z)dy \tag{A.2}$$

for various combinations of the cut orientations and their locations above or below the real axis. First let us assume

$$a<1,\ b<1,\ a+b>1. \tag{A.3}$$

For the more general integral (3.5) we have shown the integral vanishes, when the integration line does not separate off the both singularities, which is the case of the eight integrals (A.2) involving combinations $h^{\pm}h_{\pm},\ h^{\pm}h_{\mp},\ h_{\pm}h^{\pm},\ h_{\mp}h^{\pm}$. Let us calculate (A.2) when e.g. $x<z$ and the both singularities are above the real axis. Then

$$|I|=|\exp(ia\pi)I_1+I_2+\exp(-ib\pi)I_3|=0, \tag{A.4}$$

where

$$I_k=\int_{L_k}\frac{dy}{|(x-y)^a(y-z)^b|} \qquad L_1\equiv(-\infty,x),\ L_2\equiv(x,z),\ L_3\equiv(z,+\infty). \tag{A.5}$$

Using simple substitutions in the known relation

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}dx=\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)} \tag{A.6}$$

and denoting $d^{a+b-1}\equiv\left|(x-z)^{a+b-1}\right|$ one can get

$$\begin{aligned} I_1 &= \frac{1}{d^{a+b-1}}\cdot\frac{\Gamma(1-a)\Gamma(a+b-1)}{\Gamma(b)} \\ I_2 &= \frac{1}{d^{a+b-1}}\cdot\frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(2-a-b)} \\ I_3 &= \frac{1}{d^{a+b-1}}\cdot\frac{\Gamma(1-b)\Gamma(a+b-1)}{\Gamma(a)}. \end{aligned} \tag{A.7}$$

After inserting into (A.4) we get

$$\cos(a\pi)\frac{\Gamma(1-a)\Gamma(a+b-1)}{\Gamma(b)}+\frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(2-a-b)}+\cos(b\pi)\frac{\Gamma(1-b)\Gamma(a+b-1)}{\Gamma(a)}=0 \tag{A.8}$$

$$\frac{\sin(a\pi)\Gamma(1-a)}{\Gamma(b)}=\frac{\sin(b\pi)\Gamma(1-b)}{\Gamma(a)}. \tag{A.9}$$

The last identity also follows from the known formula (see e.g.[1],p.256)

$$\Gamma(\gamma)\Gamma(1-\gamma)\sin(\gamma\pi)=\pi \tag{A.10}$$

Now let us calculate (A.2) for remaining combinations $h^{\pm}h^{\pm},\ h^{\pm}h^{\mp},\ h_{\pm}h_{\pm},\ h_{\mp}h_{\pm}$. For $\tau\to 0$ it holds

$$\lim_{\tau\to 0+} h(\gamma,w)=\frac{\exp(i\pi c)}{|w^{\gamma}|}, \tag{A.11}$$

where the phases c of functions h entering the integral (A.2) are in accordance with the convention (2.6) summarized in Tab.2 This table enables to obtain the phases c_k of their products which are listed in the first three columns of Tab.3 The integrals of the all combinations summarized in the table can be expressed like (A.4),

$$I=\sum_{k=1}^{3}\exp(ic_k\pi)I_k \tag{A.12}$$

If we denote

$$G\equiv\frac{2i\pi}{d^{a+b-1}}\cdot\frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}, \tag{A.13}$$

then using identities (A.8),(A.10) the sum (A.12) can be evaluated. The results are given in the last column of Tab.3. Let us compare the corresponding rows in upper and lower part of the table. Obviously for any $x,\ z$ one can write

$$\lim_{\tau\to 0+}\int_{-\infty}^{+\infty}h^{\pm}(a,x-y)h^{\pm}(b,y-z)dy=\lim_{\tau\to 0+}\frac{2i\pi\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}h^{\pm}(a+b-1,x-z) \tag{A.14}$$

and equally for $h_{\pm}$. So far we have assumed (A.3), however using identity

$$\frac{d}{dw}h(\gamma,w)=-\gamma h(\gamma+1,w) \tag{A.15}$$

we can enlarge validity of (A.14) to any $a,\ b$

$$a+b>1,\qquad a,b\neq 0,-1,-2,-3,... \tag{A.16}$$

	$h(a,x-y)$		$h(b,x-z)$	
	$y<x$	$y>x$	$y<z$	$y>z$
h^+	0	$-a$	$-b$	0
h^-	0	$-a$	$-b$	0
h_+	$-2a$	$-a$	$-b$	$-2b$
h_-	0	a	b	0

Table 2: The phases c depending on the cut location

	$y<x$	$x<y<z$	$z<y$	I
h^+h^+	$-b$	$-a-b$	$-a$	$+\exp(-i\pi[a+b])G$
h_+h_+	$-2a-b$	$-a-b$	$-a-2b$	$-\exp(-i\pi[a+b])G$
h^+h^-	$-b$	$-a-b$	$-a$	$+\exp(-i\pi[a+b])G$
h_+h_-	$-2a+b$	$-a+b$	$-a$	$-\exp(-i\pi[a-b])G$
h^-h^+	$-b$	$-a-b$	$-a$	$+\exp(-i\pi[a+b])G$
h_-h_+	$-b$	$+a-b$	$+a-2b$	$-\exp(+i\pi[a-b])G$
h^-h^-	$-b$	$-a-b$	$-a$	$+\exp(-i\pi[a+b])G$
h_-h_-	$+b$	$+a+b$	$+a$	$-\exp(+i\pi[a+b])G$

	$y<z$	$z<y<x$	$x<y$	I
h^+h^+	$-b$	0	$-a$	$-G$
h_+h_+	$-2a-b$	$-2a-2b$	$-a-2b$	$+\exp(-2i\pi[a+b])G$
h^+h^-	$-b$	0	$-a$	$-G$
h_+h_-	$-2a+b$	$-2a$	$-a$	$+\exp(-2i\pi a)G$
h^-h^+	$-b$	0	$-a$	$-G$
h_-h_+	$-b$	$-2b$	$+a-2b$	$+\exp(-2i\pi b)G$
h^-h^-	$-b$	0	$-a$	$-G$
h_-h_-	$+b$	0	$+a$	$+G$

Table 3: The phases c_k of products $h(a,x-y)h(b,y-z)$ and resulting integral I (last column) for the case $x<z$ (upper part) and $x>z$ (lower part)

In this way we have proven the composition relation (3.12). (Note that $a = \alpha + 1,\ b = \beta + 1$).

Alternatively, composition relation can be easily proved in representation given by Eq. (2.28) for $\alpha,\ \beta < 0$. The relations

$$D_{\pm}^{\alpha+\beta}(x-z) = \int_{-\infty}^{+\infty} D_{\pm}^{\alpha}(x-y) D_{\pm}^{\beta}(y-z) dy, \qquad \alpha,\ \beta < 0 \tag{A.17}$$

after inserting from (2.28) and simple substitutions immediately follow from (A.6).

Acknowledgement. I would like to express my gratitude to P. Kolář for critical reading of the manuscript and valuable comments.

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